Worksheet 2.6 Factorizing Algebraic Expressions

Section 1 Finding Factors

Factorizing algebraic expressions is a way of turning a sum of terms into a product of smaller ones. The product is a multiplication of the factors. Sometimes it helps to look at a simpler case before venturing into the abstract. The number 48 may be written as a product in a number of different ways:

$$48 = 3 \times 16 = 4 \times 12 = 2 \times 24$$

So too can polynomials, unless of course the polynomial has no factors (in the way that the number 23 has no factors). For example:

$$x^3 - 6x^2 + 12x - 8 = (x-2)^3 = (x-2)(x-2)(x-2) = (x-2)(x^2 - 4x + 4)$$

where $(x-2)^3$ is in fully factored form.

Occasionally we can start by taking common factors out of every term in the sum. For example,

$$\begin{aligned} 3xy + 9xy^2 + 6x^2y &= 3xy(1) + 3xy(3y) + 3xy(2x) \\ &= 3xy(1 + 3y + 2x) \end{aligned}$$

Sometimes not all the terms in an expression have a common factor but you may still be able to do some factoring.

Example 1 :

$$9a^2b + 3a^2 + 5b + 5b^2a = 3a^2(3b+1) + 5b(1+ba)$$

Example 2 :

$$\begin{aligned} 10x^2 + 5x + 2xy + y &= 5x(2x+1) + y(2x+1) \qquad \text{Let } T = 2x+1 \\ &= 5xT + yT \\ &= T(5x+y) \\ &= (2x+1)(5x+y) \end{aligned}$$

Example 3 :

$$\begin{aligned} x^2 + 2xy + 5x^3 + 10x^2y &= x(x+2y) + 5x^2(x+2y) \\ &= (x+5x^2)(x+2y) \\ &= x(1+5x)(x+2y) \end{aligned}$$

Exercises:

1. Factorize the following algebraic expressions:

   (a) $6x + 24$

   (b) $8x^2 - 4x$

   (c) $6xy + 10x^2y$

   (d) $m^4 - 3m^2$

   (e) $6x^2 + 8x + 12yx$

   For the following expressions, factorize the first pair, then the second pair:

   (f) $8m^2 - 12m + 10m - 15$

   (g) $x^2 + 5x + 2x + 10$

   (h) $m^2 - 4m + 3m - 12$

   (i) $2t^2 - 4t + t - 2$

   (j) $6y^2 - 15y + 4y - 10$

Section 2 Some standard factorizations

Recall the distributive laws of section 1.10.

Example 1 :

$$\begin{aligned}(x+3)(x-3) &= x(x-3)+3(x-3)\\ &= x^2-3x+3x-9\\ &= x^2-9\\ &= x^2-3^2\end{aligned}$$

Example 2 :

$$\begin{aligned}(x+9)(x-9) &= x(x-9)+9(x-9)\\ &= x^2-9x+9x-81\\ &= x^2-81\\ &= x^2-9^2\end{aligned}$$

Notice that in each of these examples, we end up with a quantity in the form $A^2 - B^2$. In example 1, we have

$$\begin{aligned} A^2 - B^2 &= x^2 - 9 \\ &= (x+3)(x-3) \end{aligned}$$

where we have identified $A = x$ and $B = 3$. In example 2, we have

$$\begin{aligned} A^2 - B^2 &= x^2 - 81 \\ &= (x+9)(x-9) \end{aligned}$$

where we have identified $A = x$ and $B = 9$. The result that we have developed and have used in two examples is called the difference of two squares, and is written:

$$A^2 - B^2 = (A+B)(A-B)$$

The next common factorization that is important is called a perfect square. Notice that

$$\begin{aligned} (x+5)^2 &= (x+5)(x+5) \\ &= x(x+5) + 5(x+5) \\ &= x^2 + 5x + 5x + 25 \\ &= x^2 + 10x + 25 \\ &= x^2 + 2(5x) + 5^2 \end{aligned}$$

The perfect square is written as:

$$(x+a)^2 = x^2 + 2ax + a^2$$

Similarly,

$$\begin{aligned} (x-a)^2 &= (x-a)(x-a) \\ &= x(x-a) - a(x-a) \\ &= x^2 - ax - ax + a^2 \\ &= x^2 - 2ax + a^2 \end{aligned}$$

For example,

$$\begin{aligned} (x-7)^2 &= (x-7)(x-7) \\ &= x(x-7) - 7(x-7) \\ &= x^2 - 7x - 7x + 7^2 \\ &= x^2 - 14x + 49 \end{aligned}$$

Exercises:

1. Expand the following, and collect like terms:
   (a) $(x+2)(x-2)$
   (b) $(y+5)(y-5)$
   (c) $(y-6)(y+6)$
   (d) $(x+7)(x-7)$
   (e) $(2x+1)(2x-1)$
   (f) $(3m+4)(3m-4)$
   (g) $(3y+5)(3y-5)$
   (h) $(2t+7)(2t-7)$
2. Factorize the following:
   (a) x^2-16
   (b) y^2-49
   (c) x^2-25
   (d) $4x^2-25$
   (e) $16-y^2$
   (f) m^2-36
   (g) $4m^2-49$
   (h) $9m^2-16$
3. Expand the following and collect like terms:
   (a) $(x+5)(x+5)$
   (b) $(x+9)(x+9)$
   (c) $(y-2)(y-2)$
   (d) $(m-3)(m-3)$
   (e) $(2m+5)(2m+5)$
   (f) $(t+10)(t+10)$
   (g) $(y+8)^2$
   (h) $(t+6)^2$
4. Factorize the following:
   (a) y^2-6y+9
   (b) $x^2-10x+25$
   (c) $x^2+8x+16$
   (d) $x^2+20x+100$
   (e) $m^2+16m+64$
   (f) $t^2-30t+225$
   (g) $m^2-12m+36$
   (h) $t^2+18t+81$